ASYMPTOTIC DIMENSION AND BOUNDARIES OF HYPERBOLIC SPACES

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ABSTRACT. We give an example of a visual Gromov-hyperbolic metric space X with $asdim = 2$ and $dim(\partial X) = 0$.

1. Introduction

The notion of asymptotic dimension of a metric space was introduced by Gromov in [2]. It is a large scale analog of topological dimension and it is invariant by quasi-isometries. This notion has proved relevant in the context of Novikov's higher signature conjecture. Yu [7] has shown that groups of finite asymptotic dimension satisfy Novikov's conjecture. Dranishnikov ([6]) has investigated further asymptotic dimension generalizing several theorems from topological to asymptotic dimension.

In this paper we are concerned with the relationship between asymptotic dimension of a Gromov-hyperbolic space (see [1]) and the topological dimension of its boundary. Gromov in [2], sec. $1.E_1'$ sketches an argument that shows that complete simply connected manifolds M with pinched negative curvature have asymptotic dimension equal to their dimension. He observes that the same argument shows that $asdim(G) < \infty$ for G a hyperbolic group and asks whether such considerations lead further to the inequality $asdim(G) \leq dim(\partial G) + 1$.

Bonk and Schramm ([3]) have shown that if X is a Gromov-hyperbolic space of bounded growth then X embeds quasi-isometrically to the hyperbolic n-space $\mathbb{H}^n$ for some n. It follows that $asdim(X) < \infty$ (see also [8] for a proof of this). If K is any metric space one can define ([1], [3]) a hyperbolic space $Con(K)$ with $\partial Con(K) = K$. If X is a visual hyperbolic space then X is quasi-isometric to $Con(\partial X)$ (i.e. the boundary 'determines' the space). So it is natural to ask whether $asdim(X) \leq \dim(\partial X) + 1$ for visual hyperbolic spaces in general. Besides the argument sketched in [2], sec. $1.E_1'$ makes sense in this context too.

In this paper we give an example of a visual hyperbolic space X such that $asdim X = 2$ and $dim \partial X = 0$. So the inequality $asdim(X) \leq dim(\partial X) + 1$ doesn't hold for this space.

We remark finally that Gromov's question for hyperbolic group was settled in the affirmative recently by Buyalo and Lebedeva [4].

2. Preliminaries

Metric Spaces. Let (X,d) be a metric space. The *diameter* of a set B is denoted by diam(B). A *path* in X is a map $\gamma : I \to X$ where I is an interval in $\mathbb{R}$. A path

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γ joins two points x and y in X if I [a,b] and γ(a) = x , γ(b) = y . The path γ is called an infinite ray starting from x_0 if I=[0,∞) and $\gamma(0) = x_0$. A geodesic, a geodesic ray or a geodesic segment in X is an isometry $\gamma : I \to X$ where I is $\mathbb{R}$ or $[0, \infty)$ or a closed segment in $\mathbb{R}$. We use the term geodesic, geodesic ray etc for the images of γ without discrimination. On a path connected space X given two points x,y we define the path metric to be $\rho(x, y) = inf\{length(p)\}$ where the infimum is taken over all paths p that connect x and y (of course $\rho(x, y)$ might be infinite). It is easy to see that inside a ball B(x,n) of the hyperbolic plane or the euclidian plane the path metric and the usual metric coincide. A metric space (X, d) is called *geodesic metric space* if $d = \rho$ (the path metric is equal to the metric).

Hyperbolic Spaces. Let (X,d) be a metric space . Given three points x,y,z in X we define the *Gromov Product* of x and y with respect to the basepoint w to be :

$$(x|y)_w = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y))$$

A space is said to be δ- *hyperbolic* if for all x,y,z,w in X we have:

$$(x|z)_w \geq min\{(x|y)_w, (y|z)_w\} - \delta$$

A sequence of points $\{x_i\}$ in X is said to converge at infinity if:

$$\lim_{i,j\to\infty} (x_i|x_j)_w = \infty$$

Two sequences $\{x_i\}$ and $\{y_i\}$ are equivalent if:

$$\lim_{i,j\to\infty} (x_i|y_j) = \infty$$

This is an equivalence relation which does not depend on the choice of w (easy to see). The boundary ∂X of X is defined as the set of equivalence classes of sequences converging at infinity. Two sequences are 'close' if $\liminf_{i,j\to\infty}(x_i|y_j)$ is big. This defines a topology on the boundary.

The boundary of every proper hyperbolic space is a compact metric space.

If X is a geodesic hyperbolic metric space and $x_0 \in X$ then ∂X can be defined as the set of geodesic rays from x_0 where we define to rays to be equivalent if they are contained in a finite Hausdorf neighborhood of each other. We equip this with the compact open topology.

A metric d on the boundary ∂X of X is said to be *visual* if there are $x_0 \in X, a > 1$ and $c_1, c_2 > 0$ such that

$$c_1 a^{-(z,w)_{x_0}} \leq d(z, w) \leq c_2 a^{-(z,w)_{x_0}}$$

for every z,w in ∂X. The boundary of a hyperbolic space always admits a visual metric (see [1]).

A hyperbolic space X is called *visual* if for some $x_0 \in X$ there exists a $D > 0$ such that for every $x \in X$ there exists a geodesic ray r from x_0 in ∂X such that $d(x, r) \leq D$ (see more on [3]). It is easy to see that if X is visual with respect to a base point x_0 then it is visual with respect to any other base point.

Topological Dimension. A covering $\{B_i\}$ has *multiplicity* n if no more than $n+1$ sets of the covering have a non empty intersection. The *mesh* of the covering is the largest of the diameters of the B_i.

We will use in this paper the following definition of topological dimension for compact metric spaces which is equivalent to the other known definitions : A compact metric space has *dimension* $\leq n$ if and only if it has coverings of arbitrarily small mesh and order $\leq n$. (see [5])

Asymptotic Dimension. A metric space Y is said to be d - disconnected or that it has dimension 0 on the d - scale if

$$Y \ = \bigcup_{i \in I} B_i$$

such that: $sup\{diamB_i, i \in I\} = D < \infty$, dist$(B_i, B_j) \geq d \ \forall i \neq j$ where dist$(B_i, B_j) = \inf$ {dist(a,b) a$\in B_i$, b $\in B_j\}$
(Asymptotic Dimension 1). We say that a space X has asymptotic dimension n if n is the minimal number such that for every $d > 0$ we have : $X = \bigcup X_k$ for k = 1,2, ... n and all X_k are d-disconnected. We then write asdim = n

We say that a covering $\{B_i\}$ has *d - multiplicity* , k if and only if every d - ball in X meets no more than k sets B_i of the covering.A covering has *multiplicity* n if no more than n + 1 sets of the covering have one a non empty intersection. A covering $\{B_i\}\, i \in I$ is *D - bounded* if diam $(B_i) \leq D \, \forall \, i \in I$
(Asymptotic Dimension 2). We say that a space X has asdim = n if n is the minimal number such that $\forall \, d > 0$ there exists a covering of X of uniformly D - bounded sets B_i such that d - multiplicity of the covering $\leq n + 1$. The two definitions are equivalent. (see [1])

The Hyperbolic Plane. The hyperbolic plane $\mathbb{H}^2$ is a visual hyperbolic space of bounded geometry. It is easy to see that $asdim \mathbb{H}^2 = 2$ (see [2]). We will use the standard model of the hyperbolic plane given by the interior of a disk in $\mathbb{R}^2$.

3. Constructing The "COMB" Space

Let $\mathbb{H}^2$ be the hyperbolic plane and let $a_1, a_2, ...$ be geodesic rays starting from a point x_0 and extending to infinity such that the angle between a_n, a_{n+1} is $\frac{\pi}{2^n}$.

Let $S(a_n, a_{n+1})$ be the sector defined by the rays a_n, a_{n+1}. In other words $S(a_n, a_{n+1})$ is the convex closure of a_n, a_{n+1}.

Since geodesics diverge in $\mathbb{H}^2$ there is an $x \in S(a_n, a_{n+1})$ such that the ball of radius n and center x, $B(x, n)$ is contained in $S(a_n, a_{n+1})$. Let N_n be such that $B(x, n) \subset B(x_0, N_n)$. Let

$$S(a_n, a_{n+1}, N_n) = S(a_n, a_{n+1}) \cap B(x_0, N_n)$$

Let's call K_n the upper arc of $S(a_n, a_{n+1}, N_n)$, i.e.

$$K_n = S(a_n, a_{n+1}, N_n) \cap \partial B(x_0, N_n)$$

We subdivide K_n into small pieces of length between 1/2 and 1 marking the vertices. Then we consider the geodesic rays starting from x_0 to every vertex we defined and we extend them to infinity.

So we arrive at the "COMB" space which is the union of all the $S(a_n, a_{n+1}, N_n)$ together with these rays and looks like this:

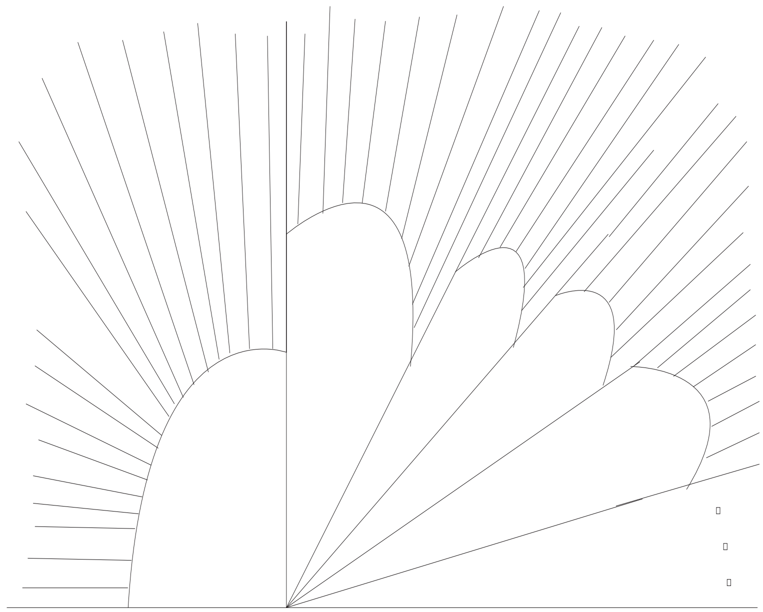

FIGURE 1. Comb Space

4. The Properties Of "COMB" Space

a) $dim(\partial X) = 0$. For every n we have that K_n is bounded. That means that we define a finite number of vertices on every K_n so we add a finite number of geodesic rays. So, all the infinite geodesic rays are countable. So ∂X is countable. Now a countable metric space has dimension 0 (see [5] page 18). So dim(∂X)=0

b) X is a hyperbolic space with the "path" metric. That is true since every pair of points of X can be joined by a path of finite length. Also let l be a closed curve of X then l is a closed curve in $\mathbb{H}^2$ and $length(l)_X \geq length(l)_{\mathbb{H}^2}$. But since $\mathbb{H}^2$ is hyperbolic we have the isoperimetric inequality $Area(l) \leq c * length(l)_{\mathbb{H}^2}$ so $Area(l) \leq c * length(l)_X$ which means that X is hyperbolic.(see [1], [9])

c) $asdim(X) = 2$. That is because X contains arbitrarily large balls $B(x, n) \subset \mathbb{H}^2$ for every $n \in \mathbb{N}$.

d) X is a visual hyperbolic space with $D = 1$ since for every x in X there exists a geodesic from x_0 to x. Let's call that g_1. If g_1 can be extended to infinity then we have nothing to prove. Let g_1 be finite ,then x must belong to a sector $S(a_n, a_{n+1}, N_n)$. We extend g_1 until it meets K_n at a point v_1. Then by the construction of X there exists an infinite geodesic r corresponding to the vertex on K_n v such that $d(v_1, v)$ is less than 1. Then obviously $d(x, r)$ is less than 1.

So X is a visual hyperbolic metric space such that that $asdimX > dim\partial X + 1$.

We remark that it is not very hard to see that X is quasi-isometrically embedded in $\mathbb{H}^2$.

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